

THE SPINLESS RELATIVISTIC WOODS–SAXON PROBLEM

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Abstract

Motivated by the observation of a recent renewal of rather strong interest in the description of bound states by (semi-)relativistic equations of motion, we revisit, for the example of the Woods–Saxon interactions, the eigenvalue problem posed by the spinless Salpeter equation and recall various elementary knowledge, considerations, and techniques that practitioners seeking solutions to this specific reduction of the Bethe–Salpeter equation may find helpful.

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1 Introduction: Semirelativistic Equations of Motion

The recent couple of years have witnessed a huge revival of interest in investigating systems of (bound-state) constituents by means of the semirelativistic equation of motion called the *spinless Salpeter equation*,¹ which can be looked at from two rather opposite points of view:

- On the one hand, it may be regarded as both straightforward and very likely simplest generalization of the nonrelativistic Schrödinger equation towards proper inclusion of relativistic kinematics by the relativistically correct expression for the kinetic energy.
- On the other hand, it is encountered along the course of *three-dimensional reductions* of the *Bethe–Salpeter formalism* allowing for relativistically covariant descriptions of bound states within quantum field theory [4–6] if assuming instantaneous interaction between and free propagation of the bound-state constituents [7] and disregarding all negative-energy contributions (which may be justified for semirelativistic and weakly bound heavy constituents) as well as the spin degrees of freedom of the constituents.²

Irrespective of its conceptual roots, the spinless Salpeter equation is an eigenvalue equation

$$H |\chi_k\rangle = E_k |\chi_k\rangle, \quad k = 0, 1, 2, \dots,$$

for the *eigenstates* $|\chi_k\rangle$ and corresponding *eigenvalues* E_k of a Hamiltonian H composed of the kinetic energies of the spin-zero particles forming the system under consideration and a static potential subsuming all the interactions of these constituents. The relativistic kinetic energy of a spin-zero particle of effective mass m and spatial momentum \mathbf{p} is represented by the notorious *square-root operator* $\sqrt{\mathbf{p}^2 + m^2}$.³ For systems consisting of just two particles, only their relative momentum \mathbf{p} and relative coordinate \mathbf{x} are relevant for the Hamiltonian:

$$H \equiv T(\mathbf{p}) + V(\mathbf{x}). \quad (1)$$

For simplicity, let these two particles have equal masses m and thus equal kinetic energies:⁴

$$H \equiv 2 \sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{x}). \quad (2)$$

The, in general, nonlocal nature of this operator renders difficult to arrive at exact analytic statements about the solutions of its eigenvalue equation. On the other hand, it is very easy to analyze the reliability of any such solution, irrespective of how it has been gained, by the relativistic generalization of the virial theorem [11, 12] or to infer upper bounds on energies.

Here, we focus to spherically symmetric central potentials, depending only on the radial coordinate $r \equiv |\mathbf{x}|$, that is, $V(\mathbf{x}) = V(r)$. One particularly prominent interaction potential is, because of its pivotal rôle for the shell model of nuclear physics, the (real) Woods–Saxon potential, a short-ranged potential, specified by three parameters: the coupling strength V_0 determining the potential’s depth, the potential’s width R , and the surface thickness a [13]:

$$V(r) = -\frac{V_0}{1 + \exp\left(\frac{r-R}{a}\right)}, \quad V_0 > 0, \quad R \geq 0, \quad a > 0. \quad (3)$$

The Hamiltonian operator (2) with the potential (3) poses what we call the semirelativistic spinless Woods–Saxon problem. Very well aware of the difficulties encountered when trying to derive *exact* analytical solutions to any such kind of problems, we formulate, by applying standard tools, the trivial boundary conditions to all corresponding *approximate* solutions.

¹Several aspects and facets of the spinless Salpeter formalism are concisely reviewed in, *e.g.*, Refs. [1–3].

²The most essential steps of this well-defined nonrelativistic reduction are sketched in, *e.g.*, Refs. [8–10].

³Throughout our analysis, we adopt, of course, natural units appropriate for particle physics: $\hbar = c = 1$.

⁴The straightforward generalization to the case of constituents with unequal masses is a trivial exercise.

2 Rigorous Upper Limit on Number of Bound States

One paramount specific of all bound-state problems is the number of possible bound states.

2.1 Nonrelativistic Kinematics: the Schrödinger Equation

In the case of the Schrödinger equation, results are abundant. For the Schrödinger operator

$$H = \frac{\mathbf{p}^2}{2\mu} + V(r) , \quad \mu > 0 ,$$

Bargmann [14] found, for the number n_ℓ of bound states with orbital angular momentum ℓ ,

$$(2\ell + 1) n_\ell \lesssim I \equiv 2\mu \int_0^\infty dr r |V_-(r)| , \quad V_-(r) \equiv -\max[0, -V(r)] .$$

Then, $n_\ell \geq 1$ implies $\ell \leq \ell_{\max} = (I-1)/2$, and the total number N of bound states satisfies

$$N = \sum_{\ell=0}^{\ell_{\max}} (2\ell + 1) n_\ell \lesssim I (\ell_{\max} + 1) = \frac{I(I+1)}{2} . \quad (4)$$

2.2 Relativistic Kinematics: the Spinless Salpeter Equation

For the spinless Salpeter equation, results on the number of bound states are sparse [15,16]; an upper limit on this quantity as easy to handle as the Bargmann limit has been proved by I. Daubechies [15]: for technical reasons, let the Hamiltonian operator $H = K + V$ acting on $L^2(\mathbb{R}^3)$ be composed of a kinetic term K that is a positive, strictly increasing, differentiable function of $|\mathbf{p}|$ only that vanishes at $|\mathbf{p}| = 0$ (which in the relativistic case can be effected by subtracting appropriate multiples of m) and rises beyond bounds with increasing $|\mathbf{p}|$ and of some potential V that is a negative smooth function of compact support, $V \in C_0^\infty(\mathbb{R}^3)$, *i.e.*, $H \equiv K(|\mathbf{p}|) + V(\mathbf{x})$, $K(|\mathbf{p}|) \geq 0$, $K(0) = 0$, $K(|\mathbf{p}|) \xrightarrow{|\mathbf{p}| \rightarrow \infty} \infty$, $V(\mathbf{x}) \leq 0$.

The total number N of bound states of such suitable operators is subject to the upper limit

$$N \leq \frac{C}{6\pi^2} \int d^3x [K^{-1}(|V(\mathbf{x})|)]^3 , \quad (5)$$

which involves a constant C depending on the kinetic energy K but not on the potential V :

$$C = \inf_{b>0} \left(\left\{ e^b \int_0^\infty \frac{dy}{y^2} e^{-by} [g(y)]^3 \right\} \left\{ b \int_0^\infty \frac{dy y}{y+1} e^{-by} \right\}^{-1} \right) , \quad g(y) \equiv \sup_{x>0} \frac{K^{-1}(xy)}{K^{-1}(x)} .$$

Specifically, subsuming one-body and equal-mass two-body *relativistic kinematics*, we have

$$K(|\mathbf{p}|) = \alpha \left(\sqrt{|\mathbf{p}|^2 + m^2} - m \right) , \quad \alpha = 1, 2 . \quad (6)$$

The constant C is independent also of α : $C = 6.074898$ for $m = 0$, $C = 14.10759$ for $m > 0$. The number of bound states of a spinless Salpeter equation with potential $V(r)$ is, at most,

$$N \leq \frac{C}{6\pi^2} \int d^3x \left[\frac{|V(r)|}{\alpha} \left(\frac{|V(r)|}{\alpha} + 2m \right) \right]^{3/2} .$$

In particular, in the case of two bound particles of equal mass ($\alpha = 2$), this inequality reads

$$N \leq \frac{C}{48\pi^2} \int d^3x [|V(r)| (|V(r)| + 4m)]^{3/2} = \frac{C}{12\pi} \int_0^\infty dr r^2 [|V(r)| (|V(r)| + 4m)]^{3/2} .$$

3 Schrödinger Upper Bounds on Energy Eigenvalues

Unspectacular upper bounds on the eigenvalues of any semirelativistic Hamiltonian (2) are provided by the eigenvalues of its nonrelativistic limit H_{NR} [3,9,10,17], since the positivity

$$\left(\sqrt{\mathbf{p}^2 + m^2} - m\right)^2 \geq 0$$

of the square of the self-adjoint operator $\sqrt{\mathbf{p}^2 + m^2} - m$ means that for any spinless particle the relativistic kinetic energy is bounded from above by the nonrelativistic kinetic energy:⁵

$$\sqrt{\mathbf{p}^2 + m^2} \leq m + \frac{\mathbf{p}^2}{2m} .$$

This operator inequality carries over to the full Hamiltonian H , involving a potential $V(\mathbf{x})$; the semirelativistic operator is bounded from above by its nonrelativistic counterpart H_{NR} :

$$H \equiv T(\mathbf{p}) + V(\mathbf{x}) \leq H_{\text{NR}} \equiv 2m + \frac{\mathbf{p}^2}{m} + V(\mathbf{x}) .$$

With this inequality at one's disposal, it is easy to convince oneself [3,9,10,17] that any pair $(E_k, E_{k,\text{NR}})$ of associated eigenvalues necessarily satisfies $E_k \leq E_{k,\text{NR}}$ for all $k = 0, 1, 2, \dots$

4 Laguerre Variational Upper Bounds to Eigenvalues

The *minimum–maximum theorem* [18–20] offers easy-to-handle tools for the localization of the *discrete* spectrum of an operator H in Hilbert space: Let H be *self-adjoint* and *bounded from below* and its *eigenvalues* E_k , defined by the eigenvalue equation $H|\chi_k\rangle = E_k|\chi_k\rangle$ for the associated eigenstates $|\chi_k\rangle$, $k = 0, 1, \dots$, be *ordered* according to $E_0 \leq E_1 \leq E_2 \leq \dots$. Then, restricting H to a d -dimensional *subspace* D_d of its domain, the eigenvalues E_k *below* the onset of the *essential* spectrum of H if counting multiplicity of degenerate states satisfy

$$E_k \leq \sup_{|\psi\rangle \in D_{k+1}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \quad \text{for all } k = 0, 1, 2, \dots .$$

An immediate consequence of this theorem is the Rayleigh–Ritz variational technique. The d eigenvalues \hat{E}_k , $k = 0, 1, \dots, d-1$, of this operator H restricted to a subspace D_d , likewise ordered according to $\hat{E}_0 \leq \hat{E}_1 \leq \dots \leq \hat{E}_{d-1}$, are *upper bounds* to the first d eigenvalues E_k :

$$E_k \leq \hat{E}_k , \quad k = 0, 1, \dots, d-1 .$$

The quality or tightness [21,22] of upper bounds found in this way may be increased by just enlarging the trial space D_d by raising its dimension; at least, this won't make things worse.

The Woods–Saxon potential (3) possesses the merit of being non-singular: the operator (2) is bounded from below. For a relativistic Coulomb problem, posed by assuming $V(\mathbf{x})$ to be the Coulomb potential $V_{\text{C}}(\mathbf{x}) \propto -1/|\mathbf{x}|$, for instance, both essential self-adjointness and boundedness from below of the operator (2) has been rigorously established by Herbst [23].

⁵Geometrically, this is just a consequence of the concavity of the square root. The nonrelativistic kinetic energy is the tangent line to the square root of the relativistic kinetic energy, at the point of contact $\mathbf{p}^2 = 0$.

Hence, we feel entitled to exploit the minimum–maximum theorem for the construction of rigorous upper bounds to the energy eigenvalues E_k of the semirelativistic Woods–Saxon Hamiltonian. By spherical symmetry, every basis function of the Hilbert space $L^2(\mathbb{R}^3)$ (*i.e.*, the Hilbert space of the square-integrable functions on the three-dimensional space \mathbb{R}^3) is a product of a radial function and a spherical harmonic $\mathcal{Y}_{\ell m}(\Omega)$ for angular momentum ℓ and projection m that depends on the solid angle Ω in the underlying representation space. We span the trial spaces D_d of the minimum–maximum theorem by an orthonormal set of basis vectors [3,9,10,24,25], involving two variational parameters: μ , with the dimension of mass, and β , being dimensionless; our basis states exhibit the compulsory normalizability as long as the variational parameters assume values in the intervals $0 < \mu < \infty$ and $-\frac{1}{2} < \beta < \infty$.

- The *configuration-space representation* $\psi_{k,\ell m}(\mathbf{x})$ of our orthonormalized basis vectors utilizes the *generalized-Laguerre orthogonal polynomials* $L_k^{(\gamma)}(x)$ for the parameter γ , defined by a power series [26,27], with orthogonality fixing γ uniquely to $\gamma = 2\ell + 2\beta$:

$$\psi_{k,\ell m}(\mathbf{x}) = \sqrt{\frac{(2\mu)^{2\ell+2\beta+1} k!}{\Gamma(2\ell+2\beta+k+1)}} |\mathbf{x}|^{\ell+\beta-1} \exp(-\mu |\mathbf{x}|) L_k^{(2\ell+2\beta)}(2\mu |\mathbf{x}|) \mathcal{Y}_{\ell m}(\Omega_{\mathbf{x}}) ,$$

$$L_k^{(\gamma)}(x) \equiv \sum_{t=0}^k (-1)^t \binom{k+\gamma}{k-t} \frac{x^t}{t!} , \quad k = 0, 1, 2, \dots ,$$

$$\int d^3x \psi_{k,\ell m}^*(\mathbf{x}) \psi_{k',\ell' m'}(\mathbf{x}) = \delta_{kk'} \delta_{\ell\ell'} \delta_{mm'} .$$

- The *momentum-space representation* $\tilde{\psi}_{k,\ell m}(\mathbf{p})$ of the orthonormalized basis vectors is found to be a series involving the hypergeometric function $F(u, v; w; z)$ [26], which, in turn, may be defined, in terms of the gamma function $\Gamma(z)$, by its Gauss power series:

$$\begin{aligned} \tilde{\psi}_{k,\ell m}(\mathbf{p}) &= \sqrt{\frac{(2\mu)^{2\ell+2\beta+1} k!}{\Gamma(2\ell+2\beta+k+1)}} \frac{(-i)^\ell |\mathbf{p}|^\ell}{2^{\ell+1/2} \Gamma(\ell + \frac{3}{2})} \\ &\times \sum_{t=0}^k \frac{(-1)^t}{t!} \binom{k+2\ell+2\beta}{k-t} \frac{\Gamma(2\ell+\beta+t+2) (2\mu)^t}{(\mathbf{p}^2 + \mu^2)^{(2\ell+\beta+t+2)/2}} \\ &\times F\left(\frac{2\ell+\beta+t+2}{2}, -\frac{\beta+t}{2}; \ell + \frac{3}{2}; \frac{\mathbf{p}^2}{\mathbf{p}^2 + \mu^2}\right) \mathcal{Y}_{\ell m}(\Omega_{\mathbf{p}}) , \end{aligned}$$

$$F(u, v; w; z) \equiv \frac{\Gamma(w)}{\Gamma(u) \Gamma(v)} \sum_{n=0}^{\infty} \frac{\Gamma(u+n) \Gamma(v+n)}{\Gamma(w+n)} \frac{z^n}{n!} ,$$

$$\int d^3p \tilde{\psi}_{k,\ell m}^*(\mathbf{p}) \tilde{\psi}_{k',\ell' m'}(\mathbf{p}) = \delta_{kk'} \delta_{\ell\ell'} \delta_{mm'} .$$

Of course, for any basis state its configuration-space and momentum-space representations $\psi_{k,\ell m}(\mathbf{x})$ and $\tilde{\psi}_{k,\ell m}(\mathbf{p})$ are related by Fourier transformations acting on $L^2(\mathbb{R}^3)$, their radial functions consequently by Fourier–Bessel transformations. The advantage of such choice of basis is the availability of both representations in *analytic* form. This enables us to evaluate the matrix elements of $T(\mathbf{p})$ in momentum space, and those of $V(\mathbf{x})$ in configuration space.

5 Approximations Enabling an Analytical Treatment

More or less recently, a hardly manageable amount of investigations aiming at approximate but to the largest reasonable extent analytical approaches to the spinless Salpeter equation have been published. Driven by the maybe easily comprehensible desire to achieve this goal by all available means, some of such analyses feel forced to refer to disputable assumptions.

The first of these approximations changes the nature of this problem entirely. Assuming sufficiently heavy bound-state constituents, the free energy is expanded nonrelativistically:

$$\sqrt{\mathbf{p}^2 + m^2} \approx m + \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{p}^4}{8m^3} + \dots$$

Disregarding in this expansion terms of higher than second order in \mathbf{p}^2/m^2 , *i.e.*, truncating this power series at the order \mathbf{p}^4/m^4 , generates the “pseudo spinless-Salpeter Hamiltonian”

$$H_p \equiv 2m + \frac{\mathbf{p}^2}{m} - \frac{\mathbf{p}^4}{4m^3} + V(\mathbf{x}) . \quad (7)$$

However, leaving aside the extremely unlikely possibility of a kind of mysterious conspiracy of the involved potential $V(\mathbf{x})$, this Hamiltonian H_p , as an operator on $L^2(\mathbb{R}^3)$, is definitely unbounded from below; as a consequence, a ground state of H_p does not exist. This is easily seen by considering the expectation value of H_p with respect to suitably chosen trial states; for instance, for the Laguerre basis state of Sec. 4 for the quantum numbers $n = \ell = m = 0$,

$$\psi_{0,00}(\mathbf{x}) = \sqrt{\frac{\mu^3}{\pi}} \exp(-\mu |\mathbf{x}|) , \quad \tilde{\psi}_{0,00}(\mathbf{p}) = \frac{\sqrt{8\mu^5}}{\pi} \frac{1}{(\mathbf{p}^2 + \mu^2)^2} , \quad \beta = 1 ,$$

the resulting expectation value $\langle H_p \rangle$ behaves as function of the variational parameter μ like

$$\langle H_p \rangle = 2m + \frac{\mu^2}{m} - \frac{5\mu^4}{4m^3} + \langle V(\mathbf{x}) \rangle \quad \Longrightarrow \quad \lim_{\mu \rightarrow \infty} \langle H_p \rangle = -\infty \quad \Longrightarrow \quad E_0 \leq -\infty .$$

Clearly, such annoying negligibility does not show up if, for some reasons, the term of order \mathbf{p}^4/m^4 enters into one’s bound-state equation with the wrong sign [28, Eq. (2)] [29, Eq. (2)]. If, however, the \mathbf{p}^4/m^4 term retains its correct sign, the eigenvalue equation of the operator H_p [30, Eq. (3)] [31, Eq. (3)] [32, Eq. (13)] [33, Eq. (4)] is to be taken with a big grain of salt. A promising remedy is the perturbative treatment of such bothersome contributions to H_p .

In order to eventually enforce *analytical solvability* of one’s *radial Schrödinger equation*, its centrifugal term with orbital quantum number ℓ , $\ell(\ell+1)/r^2$, arising from the Laplacian, is approximated [34] by an expansion in terms of a function $y(r)$ resembling one’s potential:

$$\frac{1}{r^2} \stackrel{!}{\approx} f(r) \equiv \sum_{j=0}^{P < \infty} c_j [y(r)]^j , \quad P \geq 2 .$$

Thus, in our case the function of choice for the new variable $y(r)$ is the Woods–Saxon shape

$$y(r) \equiv \frac{1}{1 + \exp\left(\frac{r-R}{a}\right)} .$$

However, the behaviour of $f(r)$ differs drastically from that of the centrifugal term; besides regularising the $1/r^2$ singularity at $r = 0$, it will *not* unavoidably approach zero for $r \rightarrow \infty$:

$$\lim_{r \rightarrow \infty} y(r) = 0 \quad \Longrightarrow \quad \lim_{r \rightarrow \infty} f(r) = c_0 < \infty .$$

In contrast to any *actual* Woods–Saxon problem, for $\ell \neq 0$ this allows also for bound states with (fake) *positive* energy eigenvalues E_k^f that are bounded from above by $E_k^f \leq \ell(\ell+1)c_0$.

6 Properties of Solutions to Woods–Saxon Problems

Hence, the way is paved for sketching our picture of the relativistic Woods–Saxon problem.

One of the prerequisites for the upper bound (5) to hold for generic kinetic terms $K(|\mathbf{p}|)$ (provided the latter comply with all assumptions necessary for proving this relation) is that the potential $V(\mathbf{x})$ is a smooth function with compact support, $V \in C_0^\infty(\mathbb{R}^3)$. However, the Woods–Saxon potential (3) does not have compact support and thus does not belong to the function space $C_0^\infty(\mathbb{R}^3)$. Fortunately, it can be shown [15] that, for relativistic kinematics as represented by the kinetic-energy operator (6), the range of validity of the upper bound (5) extends to all potentials $V(\mathbf{x})$ in $L^{3/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$. Since the Woods–Saxon potential (3) is — for finite values of all three parameters (*i.e.*, V_0 , R and a) characterizing this potential — in this class, we expect to be on the safe side when applying Eq. (5) to our Hamiltonian (2).

A trivial lower bound to the spectrum $\sigma(H)$ of any Hamiltonian H of the shape (1) with positive kinetic-energy operator $T(\mathbf{p}) \geq 0$ is provided by the infimum of its potential $V(\mathbf{x})$:

$$\sigma(H) \geq \inf_{\mathbf{x}} V(\mathbf{x}) .$$

For the Woods–Saxon potential (3), this lower energy bound is, clearly, realized in the form

$$\sigma(H) \geq \inf_r V(r) = \min_r V(r) = V(0) = -\frac{V_0}{1 + \exp(-\frac{R}{a})} \geq -V_0 . \quad (8)$$

We are aware of three studies [33,35,36] attempting to construct approximate solutions, by the analytic path recalled in Sec. 5, to the spinless Salpeter equation with Woods–Saxon potential. Out of these, Ref. [35] provides its results in a most detailed and explicit manner. Consequently, let us present the outcome of our considerations too for the numerical values adopted in the investigation of Ref. [35] for the mass m of both of the involved bound-state constituents and the parameters V_0 , R , and a determining the real Woods–Saxon potential:

$$\begin{aligned} m &= 4.76504 \text{ fm}^{-1} = 0.940271 \text{ GeV} , \\ V_0 &= 0.3431032 \text{ fm}^{-1} = 0.06770352 \text{ GeV} , \\ R &= 7.6136 \text{ fm} = 38.584 \text{ GeV}^{-1} , \\ a &= 0.65 \text{ fm} = 3.3 \text{ GeV}^{-1} . \end{aligned}$$

For this set of parameters, we compute the *upper limits* on the number of bound states that can be accommodated by the particular potential specified by this choice according to both nonrelativistic and relativistic kinematics discussed in Sec. 2, the variational upper bounds on the energy eigenvalues of the *spinless Salpeter equation*, generated by the Rayleigh–Ritz technique summarized in Sec. 4, and (with the help of an easily applicable standard routine for a straightforward numerical solution of Schrödinger equations [37]) their nonrelativistic counterparts reproduced in Sec. 3. The lower energy bound (8) applying likewise to spinless Salpeter and Schrödinger cases is found instantly: $E_0 \geq V(0) = -0.06770296 \text{ GeV} \geq -V_0$.

For both nonrelativistic and relativistic spinless Woods–Saxon problems, the maximum number N of bound states we may encounter is, trivially, *finite*: $0 \leq N < \infty$. For the choice used in Ref. [35] for the mass and potential parameters, we obtain for the number N , by the Daubechies bound in Eq. (5), $N \leq 850$ and, by the Bargmann bound in Eq. (4), $N \leq 1201$. So, for the present parameters, one has enough Woods–Saxon bound states to play around.

Table 1 presents our variational and Schrödinger upper limits to the *binding energies* of the lowest Woods–Saxon bound states with orbital angular momentum quantum number ℓ and radial quantum number n_r , defined as the number of zeros of the radial wave functions.

Table 1: Upper limits to the binding energy for the lowest-lying bound states of the spinless Salpeter equation with real Woods–Saxon potential, for the set of parameter values used in Ref. [35]: the trivial Schrödinger bounds \overline{E}_{NR} of Sec. 3 and the Laguerre bounds \overline{E} of Sec. 4. Any bound state is identified by its radial (n_r) and orbital angular momentum (ℓ) quantum numbers. For the bound-state problem investigated here, optimization of the orthogonality of the resulting eigenstates fixes the dimension d of the variational trial space D_d to $d = 25$; as a mere illustration, the variational parameters μ and β are kept fixed, $\mu = 1$ GeV, $\beta = 1$.

Bound state		Spinless Salpeter equation	Schrödinger equation
n_r	ℓ	$\overline{E}(n_r, \ell)$ [GeV]	$\overline{E}_{\text{NR}}(n_r, \ell)$ [GeV]
0	0	−0.06032	−0.06030
	1	−0.05309	−0.05305
1	0	−0.04119	−0.04108
	1	−0.02967	−0.02946
2	0	−0.01527	−0.01545
	1	−0.00233	−0.00362

The main assumption underlying every search for approximate solutions to the spinless Salpeter equation based on a *nonrelativistic* expansion of the relativistic free energy [which yields, as the eigenvalue equation of the Hamiltonian (7), a kind of pseudo spinless Salpeter equation] is that, for the given potential, all bound-state constituents are sufficiently heavy for this drastic modification to make sense. Before embarking on the detailed comparison of our insights with the findings of Ref. [35], let us estimate the extent to which this expansion is justified. Under the reasonable assumption that the lowest-lying bound state deduced by application of the variational technique sketched in Sec. 4, with radial and orbital quantum numbers $n_r = \ell = 0$, represents a passable description of the ground state of the relativistic Woods–Saxon problem, the expectation value of the first non-trivial term in this expansion of the operator $\sqrt{\mathbf{p}^2 + m^2}$ divided by the mass m , taken with respect to this bound state, is

$$\left\langle \frac{\mathbf{p}^2}{m^2} \right\rangle = 6.2 \times 10^{-3} .$$

Thus, the assumption is indeed justified: for the parameter values of Ref. [35], the system is highly nonrelativistic. As a matter of fact, a question that inevitably comes to one’s mind is whether, *for such systems*, it is worthwhile to invest all the efforts to overcome all obstacles of some pseudo spinless Salpeter treatment instead of sticking to the Schrödinger equation.

We take the liberty of regarding that coarse characterization of the eigenvalue spectrum of the relativistic Woods–Saxon problem given above as a collection of rigorous *constraints* that any solution of the spinless Salpeter equation with Woods–Saxon potential must fulfil. The latter request should constitute for, of course, all proposed solutions, but especially for approximate solutions of unclear quality or accuracy, a crucial criterion for their reliability. Following this spirit, let us look how well this overall pattern is reflected in Refs. [33,35,36].

However, even a merely cursory inspection of the claims of Ref. [35] leads to a number of observations which cast serious doubts on the reliability of the results reported by Ref. [35]:

- To begin with, one's confidence in all the assertions of Ref. [35] would be significantly strengthened if the binding-energy eigenvalues of the (only perturbatively accessible) pseudo spinless-Salpeter Hamiltonian defined in Eq. (7) [35, Table 1] and those of the corresponding Schrödinger Woods–Saxon problem [35, Table 2] would be compatible with our knowledge about crucial overall features of the spectrum of the Hamiltonian (2) with Woods–Saxon potential and its nonrelativistic counterpart. Regrettably, the authors of Ref. [35] preferred not to reveal the units of the binding-energy eigenvalues collected in their Tables 1 and 2. However, they present the adopted numerical values of their parameters in units of appropriate powers of fermi. So, in order to proceed we assume that their binding-energy eigenvalues too are quoted in units of inverse fermi. To our great amazement, we then realize that *all* binding-energy eigenvalues listed in Table 1 of Ref. [35] and a considerable fraction of binding-energy eigenvalues listed in Table 2 of Ref. [35] lie *below* our trivial lower spectral bound (8), and hence cannot be part of the spectra of the corresponding Hamiltonians. We regard this as a convincing indication that the entire content of the tables in Ref. [35] is rather far from the truth.
- Explicit analytic expressions for the binding-energy eigenvalues of the nonrelativistic Schrödinger equation with Woods–Saxon potential (3) can be found twice in Ref. [35] — relations (23) and (26) in Ref. [35] are, in fact, *identical*. However, this duplication [35, Eq. (23)], [35, Eq. (26)] cannot hide the fact that already on dimensional grounds this expression can by no means be correct: a quantity with non-zero mass dimension is obviously missing in the term which involves the quantum-number difference $n-L$. Nevertheless, the incorrect expression has apparently been adopted by the authors of Ref. [35] for computing the entries of their Table 2, rendering the table rather useless. The — within the plethora of approximations made in Ref. [35] — true expression for the nonrelativistic binding energies reads, for vanishing orbital angular momentum ℓ ,

$$E_{n,0} = -\frac{1}{m} \left(\frac{m V_0 a^2 + n^2}{2 n a} \right)^2 \quad \text{for } m \equiv m_1 = m_2 . \quad (9)$$

- In quantum theory, a preeminent criterion for the reliability of solutions to eigenvalue equations of self-adjoint operators is established by a generalization [12], to arbitrary kinematics, of the virial theorem well known from nonrelativistic quantum mechanics [38–40]. For an operator H of the form (1), this master virial theorem [12] claims that the expectation values, taken with respect to a given eigenstate $|\chi\rangle$ of the operator H , of the (momentum-space) *radial* derivative of its kinetic-energy operator $T(\mathbf{p})$ and of the (configuration-space) *radial* derivative of its interaction potential $V(\mathbf{x})$ are equal:

$$\left\langle \chi \left| \mathbf{p} \cdot \frac{\partial T}{\partial \mathbf{p}}(\mathbf{p}) \right| \chi \right\rangle = \left\langle \chi \left| \mathbf{x} \cdot \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \right| \chi \right\rangle .$$

This (universal) relation can be straightforwardly employed for checking, one by one, presumed eigenstates, or approximations to the latter, for their validity. For instance, the bound state associated to the lowest of all binding-energy upper limits in Table 1, consequently approximating the ground state, *i.e.*, $n_r = \ell = 0$, of the semirelativistic spinless Woods–Saxon problem investigated here, passes this test with flying colours:

$$\left\langle \frac{2 \mathbf{p}^2}{\sqrt{\mathbf{p}^2 + m^2}} \right\rangle = 0.0116 \text{ GeV} = \left\langle r \frac{\partial V}{\partial r}(r) \right\rangle .$$

Sadly, even the approximate Schrödinger state for the quantum numbers $n = 1, \ell = 0$ determined by Eqs. (24), (25) of Ref. [35] for the binding energy $E_{1,0} = -0.0700$ GeV computed from the corrected expression (9) misses this not too ambitious goal badly:

$$\left\langle \frac{2\mathbf{p}^2}{m} \right\rangle = 0.0379 \text{ GeV} \neq 0.0415 \text{ GeV} = \left\langle r \frac{\partial V}{\partial r}(r) \right\rangle.$$

- Section 3 of Ref. [35] offers, at the price of a certain lack of rigour, an analytical albeit approximate solution [35, Eqs. (22)–(25)] to the Schrödinger Woods–Saxon problem, deduced as the nonrelativistic limit of corresponding results [35, Eqs. (15), (16), (20)] emerging, in Sec. 2 of Ref. [35], from an unavoidably merely perturbative approach to any bound-state problem posed by the pseudo spinless-Salpeter Hamiltonian (7). To seek an analytical discussion of a given physics problem is, of course, a legitimate goal and the successful completion of such task even a highly desirable feature of proposed solutions but has to be backed by numerical verification. Particularly for Schrödinger operators a huge variety of results on their spectral decomposition has been compiled and has found its way into the textbooks. Such findings provide straightforward tools for checking the self-consistency of alleged analytical solutions; numerical procedures for solving the Schrödinger equation are abundant in the literature, cf., *e.g.*, Ref. [37]. A lot of considerations can be performed without entering or even touching the realm of mathematical physics. Let us illustrate this for the $(n = 1, \ell = 0)$ state represented by Eqs. (27), (28) of Ref. [35], with binding energy $E_{1,0} = -0.0700$ GeV from Eq. (9):

- The *expectation value* of a given operator H in Hilbert space with respect to any of its eigenstates $|\chi_k\rangle$ is necessarily identical to its corresponding *eigenvalue* E_k :

$$H |\chi_k\rangle = E_k |\chi_k\rangle \quad \Longrightarrow \quad \frac{\langle \chi_k | H | \chi_k \rangle}{\langle \chi_k | \chi_k \rangle} = E_k, \quad k = 0, 1, 2, \dots$$

In the meantime without surprise, such triviality is not satisfied by our test case:

$$\left\langle \frac{\mathbf{p}^2}{m} + V(r) \right\rangle = -0.0377 \text{ GeV} \neq -0.0700 \text{ GeV} = E_{1,0}.$$

Irrespective of an interpretation of Eq. (21) of Ref. [35] from the physics point of view as a quantum-mechanical Schrödinger equation, Eqs. (23)–(25) of Ref. [35] provide a rather poor approximation to the solution of this differential equation.

- Even if leaving aside the restrictions imposed by requiring self-adjointness of the Schrödinger operator H on $L^2(\mathbb{R}^3)$ whose eigenvalue equation yields the *reduced* radial Schrödinger equation recalled in Ref. [35] as Eq. (21), simple inspection of the behaviour of the position-space *reduced* radial wave functions $\psi_{n,\ell}(r)$ solving the bound-state equation forces all its solutions to vanish at spatial origin $r = 0$:

$$\lim_{r \rightarrow 0} \psi_{n,\ell}(r) \stackrel{!}{=} 0.$$

Easy to guess, this property is not shared by all solutions in Eq. (24) of Ref. [35]:

$$|\psi_{1,0}(0)| = 0.06879 \text{ GeV}^{1/2} \neq 0.$$

As the Schrödinger case forms the nonrelativistic endpoint of the analysis in Ref. [35], all this does not lend too much trust in the results of the procedures used by Ref. [35].

In Ref. [36], the emerging energy eigenvalues are left encoded in an implicit relation and still await their numerical disclosure [36, Eq. (19)]; hence, there is nothing to compare with.

In Ref. [33], numerical values of the binding energies of a few bound states are explicitly given for three different sets of mass and potential parameters [33, Table 1]. For all of these sets, the expectation values of \mathbf{p}^2/m^2 over our variational ground state fall into the interval

$$6.9 \times 10^{-3} \leq \left\langle \frac{\mathbf{p}^2}{m^2} \right\rangle \leq 9.3 \times 10^{-2} ;$$

so, all three systems are also sufficiently nonrelativistic to provide some justification for the expansion of that square root of the relativistic kinetic-energy operator in powers of \mathbf{p}^2/m^2 . All the binding-energy values provided in Ref. [33] are larger than the lower spectral bound (8), $V(0) \gtrsim -V_0$. The question whether further states satisfy this bound is left unanswered.

7 Summary and Conclusions

As a kind of comment on the current quest for an analytic approach to the spinless Salpeter equation, we compiled in this brief note some general statements of quantum theory related to any such enterprise and applied these insights to the case of the Woods–Saxon potential. Although such techniques do not immediately lead to the exact solutions, they nevertheless constitute invaluable tools to scrutinize approximate solutions for their significance. In this way, these (not excessively complicated) considerations may be exploited to systematically separate wheat from chaff, as has been demonstrated for the findings presented in Ref. [35].

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